PART TWO
PROBABILITY

I. Counting

In the study of "what is possible," there are essentially two kinds of problems. First is the problem of listing everything that can happen in a given situation, and there is the problem of determining how many different things can happen (without actually constructing a complete list).

The second kind of problem is especially important, because in many cases, we really do not need a complete list, and hence, can save ourselves a great deal of work.

A. Generalized Multiplication Rule

If an operation can be performed in n_1 ways, and if for each of these, a second operation can be performed in n_2 ways, and for each of the first two a third operation can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed in $n_1 * n_2 * n_3 * \ldots * n_k$ ways.

Tree diagram – an outline showing all the possibilities for a given number of operations.

Examples:

1. Students at a private liberal arts college are classified as being freshmen, sophomores, juniors or seniors and also according to whether they are male or female. Find the total number of possible classifications for the students in this college.
2. A drug for the relief of asthma can be purchased from 5 different manufacturers, in liquid, tablet or capsule form, all from which come in regular or extra strength. In how many ways can a doctor prescribe a drug for a patient suffering from asthma? Ans: 30 ways

3. A test consists of 10 multiple-choice questions, with each question having four possible answers. In how many ways can a student
   a. check off one answer to each question? Ans: 1,048,576 ways
   b. check off one answer to each question, and get all answers wrong? Ans: 59,049 ways
   c. check off one answer to each question and get all answers correctly?

B. **Permutations** – an arrangement of all or part of a set of objects
   - grouping of things where arrangement is important

Theorems:

1) The number of permutations of n distinct objects taken all together is $_nP_n = n!$

Example 4: How many distinct permutations can be made from the letters of the word COLUMNS? Ans: 5040

How many of these permutations start with the letter M? Ans: 720

2) The number of permutations of n distinct objects taken r at a time is

$$_nP_r = \frac{n!}{(n-r)!}$$

Example 5: In a regional spelling bee, the 8 finalists consist of 3 boys and 5 girls. Find the number of possible orders at the conclusion of the contest for a) all 8 finalists; b) the first 3 positions. Ans: 40,320; 336

3) *Circular permutation* – The number of n distinct objects arranged in a circle is $(n – 1)!$

Example 6: In how many ways can 5 different trees be planted in a circle? Ans: 24 ways

4) If among n objects r_1 are identical, another r_2 are identical and the remaining (if any) are all distinct, the number of permutations of these n objects taken all together is

$$\frac{n!}{r_1!r_2!}$$

Examples:

7. A college plays 12 football games during a season. In how many ways can the team end the season with 7 wins, 3 losses and 2 ties? Ans: 7920

8. In how many ways can 3 oaks, 4 pines and 2 maples be arranged along a property line if one does not distinguish between trees of the same kind? Ans: 1260 ways

C. **Combinations** – unordered arrangement of set of objects

- grouping of objects where arrangement is NOT important

The number of combinations of *n* distinct objects taken *r* at a time is

$$_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Rule for binomial coefficients

$$\binom{n}{r} = \binom{n}{n-r} \qquad \text{for } r = 0, 1, 2, \ldots, n$$

Examples:

9. From 4 red, 5 green and 6 yellow apples, how many selections are possible if 3 of each color are to be selected? Ans: 800 selections

10. From a group of 4 men and 5 women, how many committee of size 3 are possible a) with no restrictions; b) with 1 man and 2 women; c) with 2 men and 1 woman if a certain man must be on the committee? Ans: 84, 40, 15.

11. A shipment of 12 television sets contains 3 defective sets. In how many ways can a hotel purchase 5 of these sets and receive at least 2 of the defective sets?

Exercises:

1. In a research study of women who buy mutual fund shares, women interviewed were classified into seven categories of income, four categories of investment objectives, five categories of place of residence, and two categories of occupational status. In how many ways can a mutual fund buyer be classified? Ans: 280 ways

2. a. In how many ways can 6 people be lined up to get on a bus?

b. If certain 3 persons insist on following each other, how many ways are possible?

c. If certain 2 persons refuse to follow each other, how many ways are possible? Ans: 720, 144, 480

3. a. How many 3-digit numbers can be formed from the digits 0, 1, 2, 3, 4, 5 and 6, if each digit can be used only once?

b. How many of these are odd numbers?

c. How many of these are greater than 330? Ans: 180, 75, 105

4. Nine people are going on a skiing trip in 3 cars that will hold 2, 4 and 5 passengers, respectively. In how many ways is it possible to transport the 9 people to the ski lodge using all cars? Ans: 4,410 ways

5. In how many ways can a customer purchase one or more of the following pieces of gold jewelry: a necklace, a bracelet, a ring and an anklet?

6. Find the number of ways in 6 teachers can be assigned to 4 sections of an introductory psychology course if no teacher is assigned to more than one section? Ans: 360

II. Probability Fundamentals

The most common way of measuring the uncertainties connected with events is to assign them *PROBABILITIES.*

A. Classical Probability Concept

If there are *n* equally likely possibilities, one of which must occur and *s* are regarded as favorable or a "success," then the probability of a "success" is given by the ratio *s/n*.

Examples:

12. A pair of dice is tossed. Find the probability of getting a) a total of 8; b) at most a total of 5.

13. In a poker hand consisting of 5 cards, find the probability of holding a) 3 aces; b) 4 hearts and 1 club Ans: 0.00174, 0.003576

14. If each coded item in a catalog begins with 3 distinct letters followed by 4 distinct nonzero digits, find the probability of randomly selecting one of these coded items with the first letter a vowel and the last digit even. Ans: 0.08547

B. Frequency Interpretation of Probability

The probability of an event (happening or outcome) is the proportion of the time that the events of the same events will occur in the long run.

Example 15: If 687 of 1854 freshmen who entered a men's college (over a number of years) dropped out before the end of their freshmen year, what is the probability that a freshman entering this college will drop out before the end of his freshman year? Ans: 0.37

Law of Large Numbers

If a situation, trial, or experiment is repeated again and again, the proportion of successes will tend to approach the probability that any outcome will be a success.

C. Probabilities Based on Personal or Subjective Evaluations

Such probabilities express the strength of one's belief with regard to the uncertainties that are involved, and they apply especially when there is little or no direct evidence, so that there is no choice but to consider indirect information, "educated guesses," and other subjective factors.

Some Rules of Probability

D. Sample Spaces and Events

Sample Space – set of all possible outcomes of a statistical experiment

Experiment - a process that generates data

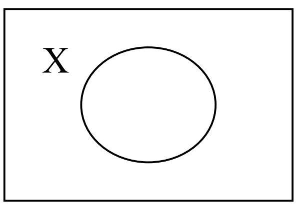
Sample spaces are described by the tabular method or the rule method. Each outcome in a sample space is called an element

Example 16: List the elements of each of the following sample spaces:

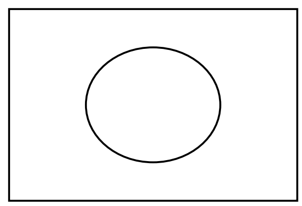
a) the set of integers between 1 and 50 divisible by 5;
b) the set $S = \{x \mid x^2 + 5x - 24 = 0\}$;
c) the set $S = \{x \mid 2x - 4 \geq 0 \text{ and } x < 1\}$.

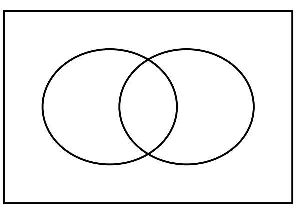
Event – subset of a sample space
Complement of an event A (denoted by A′) – the event which consists of all elements of the sample space that are not contained in A.
Intersection of two events A and B (denoted by A ∩ B) – the event which consists of all the elements contained in BOTH A and B.
Union of two events A and B (denoted by A ∪ B) – the event which consists of all the elements EITHER in event A or in event B or BOTH.
Mutually exclusive events – events having no elements in common which mean they cannot both occur at the same time.
Venn diagram – a graphical illustration showing relationships among sample spaces and events.

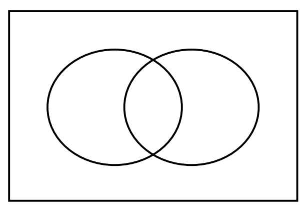
Venn Diagrams:



Event X


Event X′

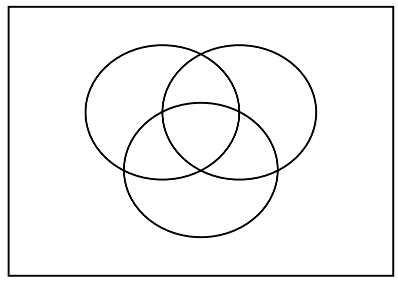

Event X ∩ Y


Event X ∪ Y

Examples:

17. Consider the sample space S = {copper, sodium, nitrogen, potassium, uranium, oxygen, zinc} and the events A = {copper, sodium, zinc}, B = {sodium, nitrogen, potassium}, C = {oxygen}. List the elements of the sets corresponding to the following events: a) A′; b) A ∪ C; c) (A ∩ B′) ∪ C′; d) B′ ∩ C′; e) A ∩ B ∩ C; f) (A′ ∪ B) ∩ (A′ ∩ C)

18. If X is the event that hamburgers will be served at the company picnic, Y is the event that beer will be served, and Z is the event that watermelon will be served, express in words the events which are represented by the following regions of the shown Venn diagram: a) region 3; b) regions 1 and 2 together; c) regions 4, 6, 7 and 8 together.



19. With reference to the example, what region or combination of regions represent that events that: a) hamburgers will not be served; b) hamburgers will be served but watermelon will not; c) beer and watermelon will both be served; d) neither beer nor watermelon will be served?

E. Some Basic Rules

Postulate 1: The probability of any event is a POSITIVE real number or zero; symbolically, $P(A) \geq 0$ for any event A.

Postulate 2: The probability of any sample space is equal to 1; symbolically, $P(S) = 1$ for any sample space.

Postulate 3: If two events are mutually exclusive, that one or the other will occur equals the sum of their probabilities. Symbolically, $P(A \cup B) = P(A) + P(B)$ for any two mutually exclusive events A and B.

Further Rules of Probability

1. There cannot be more favorable outcomes than there are outcomes; i.e., an event cannot occur more than 100 percent of the time. $P(A) \leq 1$ for any event A.
2. When an event cannot occur, there is no favorable outcome and that the probability is zero.

3. If A and A′ are complementary events, then P (A) +P (A′) = 1.
4. The probability of any event A is given by the sum of the probabilities of the individual outcomes comprising A.

General Addition Rule: Whether A and B are mutually exclusive or not:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Examples:

20. The probabilities that a service station will pump gas into 0, 1, 2, 3, 4 or 5 or more cars during a certain 30-minute period are 0.03, 0.18, 0.24, 0.28, 0.17 and 0.10. Find the probability that in this 30-minute period a) more than 2 cars receive gas; b) at most 4 cars receive gas; c) 4 or more cars receive gas. Ans: 0.55, 0.90, 0.27

21. In a certain federal prison, it is known that 2/3 of the inmates are under 25 years of age. It is also known that 3/5 of the inmates are male and that 5/8 of the inmates are female or 25 years of age or older. What is the probability that a prisoner selected at random from this prison is female and at least 25 years old? Ans: 13/120

22. The probabilities that in any given year a teenage girl will attend a professional football game, a professional tennis match or both are 0.37, 0.13 and 0.10. Determine the probability that in any year a teenage girl will attend a) a professional football game but no professional tennis match; b) a professional football game and/or a professional tennis match; c) neither a professional football game nor a professional tennis match. Ans: 0.27, 0.40, 0.60

23. Suppose that in a senior college class of 500 students, it is found that 210 smoke, 258 drink alcoholic beverages, 216 eat between meals, 122 smoke and drink alcoholic beverages, 83 eat between meals and drink alcoholic beverages, 97 smoke and eat between meals and 52 engage in all three of these bad health practices. If a member of this senior class is selected at random, find the probability that the student a) smokes but does not drink alcoholic beverages; b) eats between meals and drinks alcoholic beverages but does not smoke; c) neither smokes nor eats between meals. Ans: 88/500; 31/500, 171/500.

F. Conditional Probability

If P (A) is not equal to zero, then the conditional probability of B relative to A, namely, the probability of B given A is

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

Independent Events

Two events are said to be independent if the occurrence of one does not affect the occurrence of the other.

Two events are independent if and only if:

$$P(B/A) = P(B) \text{ and } P(A/B) = P(A)$$

Multiplicative Rules

The probability that two events will both occur is the product of the probability that one of the events will occur and the conditional probability that the other event will occur given that the first event has occurred (occurs, or will occur)

$$P(A \cap B) = P(A) \bullet P(B/A) = P(B) \bullet P(A/B)$$

If the events are independent:

$$P(A \cap B) = P(A) \bullet P(B)$$

Examples:

24. An allergist claims that 40% of the patients she tests are allergic to some type of weed. What is the probability that a) exactly 3 of her next 4 patients are allergic to weeds; b) none of her next 4 patients are allergic to weeds? Ans: 0.1536; 0.1296.

25. A random sample of 200 students are classified below according to sex and level of education attained:

Education	Male	Female
Elementary	38	45
Secondary	28	50
Tertiary	22	17

If a person is picked at random from this group, find the probability that a) the person is a male, given that the person has secondary education; b) the person does not have a college degree, given that the person is a female. Ans: 14/39; 95/112

26. Two cards are drawn in succession from a deck without replacement. What is the probability that a) both cards are red; b) both cards are greater than 3 but less than 8? Ans: 25/102; 20/221

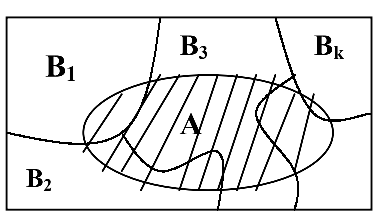
27. For married couples living in a certain city suburb, the probability that the husband will vote on a bond referendum is 0.21, the probability that the wife will vote in the referendum is 0.28 and the probability that both the husband and wife will vote is 0.15. What is the probability that a) at least one member of a married couple will vote; b) a wife will vote, given that her husband will vote; c) a husband will vote, given that his wife does not vote? Ans: 0.34; 0.714; 0.0833

28. A real estate agent has 8 master keys to open several new homes. Only one master key will open any given house. If 40% of these homes are usually left unlocked, what is the probability that the real estate agent can get into a specific home if the agent selects 3 master keys before leaving the office?

G. Bayes' Rule

Theorem of Total Probability or the Rule of Elimination (where the sample space is partitioned k subsets) – states that if the events $B_1, B_2, \ldots, B_K$ constitute a partition in the sample space S such that $P(B_I) \neq 0$ for i = 1, 2, . . .,k, then for any event of S:

$$P(A) = \sum_{i=1}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(B_i) \bullet P(A/B_i)$$



Partitioning the sample space

Bayes' Rule: If the events $B_1, B_2, \ldots, B_k$ constitute a partition in the sample space S where P (B_i) ≠ 0 for i = 1, 2, . . ., k, then for any event A in S such that P (A) ≠ 0

$$P(B_r / A) = \frac{P(B_r \cap A)}{P(A)} = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_r) * P(A/B_r)}{\sum_{i=1}^{k} P(B_i) * P(A/B_i)}$$

Examples:

29. A large industrial firm uses 3 local motels to provide overnight accommodations for its clients. From past experience, it is known that 20% of the clients are assigned rooms at the Ramada Inn, 50% at the Sheraton and 30% at the Lakeview Motor Lodge. If the plumbing is faulty in 5% of the rooms at the Ramada Inn, in 4% of the rooms at the Sheraton and in 8% of the rooms at the Lakeview Motor Lodge, what is the probability that a) a client will be assigned a room with faulty plumbing; b) a person with a room having faulty plumbing was accommodations at the Lakeview Motor Lodge? Ans: 0.054; 0.444

30. A computer software firm maintains a telephone hotline service for its customers. The firm finds that 48% of the calls involve questions about the application of the software, 38% involve issues of incompatibility with the hardware, and 14% involve the inability to install the software on the user's machine. These three categories of problems can be resolved with probabilities 0.90, 0.15 and 0.80 respectively. a) Find the probability that a call to the hotline involves a problem that can be resolved. b) If a call involves a problem that cannot be resolved, find the probability that this problem concerned incompatibility with the hardware. Ans: 0.601, 0.81.

31. A truth serum given to a suspect is known to be 90% reliable when the person is guilty and 99% reliable when the person is innocent. In other words, 10% of the guilty are judged innocent by the serum and 1% of the innocent are judged guilty. If the suspect was selected from a group of suspects of which only 5% have ever committed a crime, and the serum indicates that he is guilty, what is the probability that he is innocent?

III. Probability Distributions

A. Random Variables

A random variable is a function that associates a real number with each element in the sample space.

Example 32: Let W be a random variable giving the number of heads minus the number of tails in three tosses of a coin. List the elements of the sample space S for the three tosses of the coin and to each point assign a value *w* of W.

Types of Random Variables

1. *Discrete random variable* – data whose set of possible outcomes are countable or can represented as whole numbers.

Examples: number of automobile accidents per year, number of defective items per year, number of eggs laid each month by a hen.

2. *Continuous Random Variables* – random variables that can take on values on a continuous scale.

Examples: length of time to play a tennis match, weight of grain produced per acre, amount of milk produced by a particular cow.

B. Probability Distributions

A probability distribution is a correspondence which assigns probabilities to the values of a random variable.

Rules concerning probability distributions:

1.Since the values of a probability distribution are probabilities, they must be numbers on the interval from 0 to 1.

2. Since a random variable has to take on one of its values, the sum of values of a probability distribution must be equal to 1.

Examples:

33. A shipment of 7 television sets contains 2 defective sets. A hotel makes a random purchase of 3 of these sets. If *x* is the number of defective sets purchased by the hotel, find the probability distribution of X. Express the results graphically as a probability histogram.

34. Find the probability distribution for the number of jazz records when 4 records are selected at random from a collection consisting of 5 jazz records, 2 classical records and 3 polka records. Express the results as a formula.

C. Mean and Standard Deviation of a Probability Distribution

Mean of a probability distribution

If a random variable takes on the values $x_1, x_2, \ldots, x_k$ with the probabilities $f(x_1), f(x_2), \ldots, f(x_k)$, its *expected value* (or its mathematical expectation or mean) is given by the quantity

$$x_1 * f(x_1) + x_2 * f(x_2) + \ldots + x_k * f(x_k)$$

and is called the *mean of a random variable* or the *mean of its probability distribution.*

$$\boxed{\mu = \sum x * f(x)}$$

Standard deviation of a probability distribution

The standard deviation of a probability distribution is defined as the square root of the expected value of the squared deviation from the mean, that is,

$$\boxed{\sigma = \sqrt{\sum (x - \mu)^2 * f(x)}}$$

Example 35: Suppose that the probabilities are 0.4, 0.3, 0.2 and 0.1, respectively, that 0, 1, 2, or 3 power failures will hit a certain subdivision in any given year. Find the mean and variance of the random variable X representing the number of power failures hitting this subdivision.

Ans: $\mu = 1$, $\sigma^2 = 1$

D. Binomial Distribution

There are many applied problems in which we are interested in the probability that an event will occur "x times in n trials." Here, we are interested in the probability of getting *x successes* and *n-x failures* in *n trials.*

Assumptions:

1. The number of trials is fixed.
2. The probability of success is the same for each trial.
3. The trials are all independent.

The probability of getting *x* successes in *n* independent trials is

$$f(x) = \binom{n}{x} p^x q^{n-x} \quad \text{for } x = 0, 1, 2, \ldots, n$$

where p is the constant probability of success for each trial and q is the probability of failure, equal to 1-p.

Examples:

36. A nationwide survey of seniors by a certain university reveals that almost 70% disapprove of daily pot smoking according to a given report in 1980. If 12 seniors are selected at random and asked their opinion, find the probability that the number who disapprove of smoking pot daily is a) anywhere from 7 to 9; b) at most 5; c) not less than 8. Ans: 0.6294, 0.0386, 0.7237.

37. A traffic control engineer reports that 75% of the vehicles passing through a checkpoint are from within the state. What is the probability that fewer than 4 of the next 9 vehicles are from out of the state?
Ans: 0.8343

38. A survey of the residents in a U.S. city showed that 20% preferred a white telephone over any other color available. What is the probability that more than half of the next 20 telephones installed in this city will be white? Ans: 0.0006

E. Geometric Distribution

In some situations where otherwise the binomial distribution applies, we are interested in the probability that the FIRST success will occur on a given trial. For this to happen on the x^{th} trail, it must be preceded by x – 1 failures for which the probability is q^{x-1}, and it follows that the probability that the first success will occur on the x^{th} trial is

$$f(x) = pq^{x-1} \quad \text{for } x = 1, 2, 3, \ldots$$

This distribution is called the *geometric distribution*.

Examples:

39. The probability that a student pilot passes the written exam for his private pilot's license is 0.7. Find the probability that the person passes the test a) on the third try; b) before the fourth try. Ans:0.0630; 0.9730

40. Three people toss a coin and the odd man pays for the coffee. If the coins all turn up the same, they are tossed again. Find the probability that fewer than 4 tosses are needed. Ans: 63/64

F. Negative Binomial Distribution

If repeated independent trials can result in a success with probability *p* and a failure *q*, then the probability that the k^{th} success will occur on the x^{th} trail is

$$f(x) = \binom{x-1}{k-1} p^k q^{x-k} \quad x = k, k+1, k+2, \ldots$$

Examples:

41. A scientist inoculates several mice, one at a time, with a disease germ until he finds 2 that have contracted the disease. If the probability of contracting the disease is 1/6, what is the probability that 8 mice are required? Ans: 0.0651

42. Suppose that the probability is 0.8 that any given person will believe a tale about the transgressions of a famous actress. What is the probability that a) the sixth person to hear this tale is the fourth one to believe it; b) the third person to hear this tale is the first one to believe it?
Ans: 0.1638, 0.032

G. Multinomial Distribution

An important generalization of the binomial distribution arises when there are more than two possible outcomes for each trial, the probabilities of the various outcomes remain the same for each trial and the trials are independent.

If there are k possible outcomes for each trial and their probabilities are $p_1, p_2, \ldots, p_k$, it can be shown that the probability of x_1 outcomes of the first kine, x_2 outcomes of the second kind, and x_k outcomes of the k^{th} kind in *n* trials is given by

$$f(x_1, x_2, ..., x_n) = \frac{n!}{x_1! x_2! ... x_n!} p_1^{x_1} p_2^{x_2} ... p_k^{x_k}$$

Examples:

43. The probabilities are 0.40, 0.20, 0.30 and 0.10, respectively, that a delegate to a certain convention arrived by air, bus, automobile or train. What is the probability that among 9 delegates randomly selected at this convention, 3 arrived by air, 3 arrived by bus, 1 arrived by automobile and 2 arrived by train? Ans: 0.0077

44. The surface of a circular dartboard has a small center circle called the bull's-eye and 20 pie-shaped regions numbered from 1 to 20. Each of the pie-shaped regions is further subdivided into three parts such that a person throwing a dart that lands on a specified number scores the value of the number, double the number or triple the number, depending on which of the three parts the dart falls. If the person hits the bull's-eye with probability 0.01, hits a double with probability 0.10, hits a triple with probability 0.05, and misses the dartboard with probability 0.02, what is the probability that 7 throws will result in no bull's-eye, no triples, a double twice and a complete miss once? Ans: 0.0095

H. Hypergeometric Distribution

The hypergeometric distribution does not require independence and is based on *sampling without replacement*.

To generalize the method to use, suppose that *n* objects are to be chosen from a set of *a* objects of one kind (successes), and *b* objects of another kind (failures), the selection is without replacement and we are interested in getting *x* successes and *n* – *x* failures. For sampling without replacement, the probability of "*x* successes in *n* trials" is

$$f(x) = \frac{\binom{a}{x}\binom{b}{n-x}}{\binom{a+b}{n}} \qquad \text{for } x = 0, 1, 2 \ldots \text{ or } n$$

where *x* cannot exceed *a* and *n* – *x* cannot exceed *b*.

45. If 7 cards are dealt from an ordinary deck of 52 playing cards, what is the probability that a) exactly two of them will be face cards; b) at least one of them will be a queen? Ans: 0.3246, 0.44964
46. To avoid detection at customs, a traveler has placed 6 narcotic tablets in a bottle containing 9 vitamin pills that are similar in appearance. If the customs official selects 3 of the tablets at random for analysis, what is the probability that the traveler will be arrested for illegal possession of narcotics? Ans: 0.8154

I. Multivariate Hypergeometric Distribution

Suppose that *n* objects are to be chosen without replacement from a group consisting of a_1 of type 1, a_2 of type 2, . . . , a_k of type k. The probability of obtaining *in the sample* x_1 of type 1, x_2 of type 2, . . ., a_k of type k is given by

$$f(x_1, x_2, \ldots, x_k) = \frac{\binom{a_1}{x_1}\binom{a_2}{x_2}\cdots\binom{a_k}{x_k}}{\binom{a_1 + a_2 + \ldots + a_k}{n}}$$

In using this formula, it is assumed that there are k different types of objects and that $x_1 + x_2 + \ldots + x_k = n$.

Examples:
47. A car rental agency at a local airport has available 5 Fords, 7 Chevrolets, 4 Dodges, 3 Hondas, and 4 Toyotas. If the agency randomly selects 9 of these cars to chauffeur delegates from the airport to the downtown convention center, find the probability that 2 Fords, 3 Chevrolets, 1 Dodge, 1 Honda and 2 Toyotas are used. Ans: 0.0308
48. An urn contains 3 green balls, 2 blue balls and 4 red balls. In a random sample of 5 balls, find the probability that both blue balls and at least one red ball are selected. Ans: 0.2698

J. Poisson Distribution

Poisson experiment – experiment yielding number of outcomes occurring during a given time interval or outcomes in a specified region.

The probability of getting x successes is determined by

$$\boxed{f(x) = \frac{e^{-\mu}\mu^{x}}{x!}} \quad \text{for } x = 0, 1, 2, \ldots$$

where μ is the average number of successes

Examples:

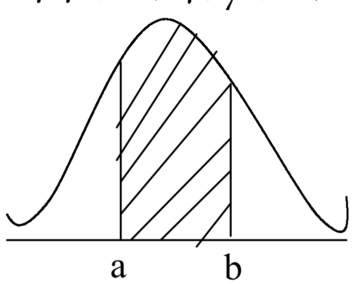
49. A certain area of the eastern United states is, on the average, hit by 6 hurricanes a year. Find the probability that in a given year, this area will be hit by a) fewer than 4 hurricanes; b) anywhere from 6 to 8 hurricanes. Ans: 0.1512, 0.4015
50. Suppose that on the average 1 person in 1000 makes a numerical error in preparing his or her ITR. If 10000 forms are selected, find the probability that 7 forms will be in error. Ans: 0.0901
51. The average number of field mice per acre in a given wheat field is estimated to be 12. Find the probability that fewer than 7 field mice are found a) on any given acre; b) on 2 of the next 3 acres inspected.

IV. Normal Distribution

A. Continuous Distribution

Continuous sample spaces arise whenever we deal with quantities that are measured on a continuous scale. In cases where there exist continuums of possibilities, in practice, we are interested with probabilities associated with intervals or regions, not individual points on a sample space.

In the continuous case, probabilities are represented by areas under continuous curves. Graphs of continuous curves are called *probability densities* or *continuous distributions*. A probability density is characterized by the following:



The area under the curve between any two values *a* and *b* gives the probability that a random variable having the continuous distribution will take on a value on the interval from *a* to *b*.

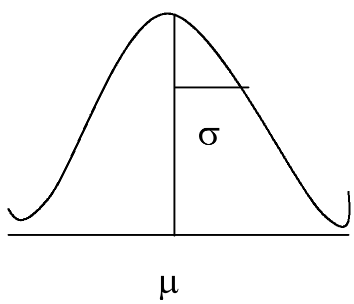
It follows that the total area under the curve is always equal to 1.

Note: In the continuous scale, the probability is <u>zero</u> that a random variable will take on any particular value.

B. Normal Distribution

The most important continuous distribution in the field of statistics is the *normal distribution*.

It was observed that discrepancies among repeated measurements of the same physical quantity displayed a surprising degree of regularity. The distribution of the discrepancies could be closely approximated by a certain curve, known as the "normal curve of errors." The mathematical equation for this type of curve is



$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

The graph of a normal distribution is a bell-shaped curve that extends indefinitely in both directions. Fortunately, it is seldom necessary to extend the tails of a normal distribution very far because the area under the curve more than 4 or 5 standard deviations away from the mean is negligible for most practical purposes.

An important feature of the normal distribution is that they depend on μ and σ, meaning that there is one and only one normal distribution with a given mean μ and a standard deviation σ.

As it is physically impossible to tabulate areas for different μ and σ, we can only tabulate areas for normal distribution with $\mu = 0$ and $\sigma = 1$ called the *standard normal distribution*. Then, we obtain areas under any normal curve by performing the change of scale, which converts any unit of measurement into a standard unit or standard score (z-score) by

$z = (x - \mu) / \sigma$ (then use Table A.3)

Note: Although the normal distribution applies to continuous random variables, it is often used to approximate distributions of discrete random variables. To do this, we must use the <u>continuity correction</u>.

Examples:

52. Given a standard normal distribution, find the area under the curve which lies a) to the left of $z = -1.39$; b) to the right of $z = 1.96$; c) between $z = -0.48$ and $z = 1.74$. Ans: 0.0823; 0.0250; 0.6435.

53. Given a normal distribution with $\mu = 30$ and $\sigma = 6$, find: a) the normal curve area to the right of $x = 17$; b) the normal curve area between $x = 32$ and $x = 41$; c) the value of x that has 80% of the normal curve area to the left; d) the value of x that has 5% of the curve area to the right.
Ans: 0.9850, 0.3371, 35.04, 39.87.

54. The loaves of rye bread distributed to local stores by a certain bakery have an average length of 30 cm and a standard deviation of 2 cm. Assuming that the lengths are normally distributed, what percentage of the loaves are a) longer than 31.7 cm; b) between 29.3 and 33.5 cm in length; c) shorter than 25.5 cm? Ans: 19.77%, 59.67%, 1.22%.

55. The finished inside diameter of a piston ring is normally distributed with a mean of 10 cm and a standard deviation of 0.03 cm. a) What proportion of the rings will have inside diameters exceeding 10.075 cm? b) What is the probability that a piston ring will have an inside diameter between 9.97 and 10.03 cm? c) Below what value of inside diameter will 15%of the piston rings fall? Ans: 0.0062, 0.6826, 9.969 cm.

56. The weights of a large number of miniature poodles are approximately normally distributed with a mean of 8 kg and a standard deviation of 0.9 kg. If measurements are recorded to the nearest tenth of a kilogram, find the fraction of these poodles with weights a) over 9.5 kg; b) at most 8.6 kg; c) between 7.3 and 9.1 kg., inclusive. Ans: 0.0427, 0.7642, 0.6964.

57. The number of complaints received by the complaint department of a department store per day is a random variable that has approximately the normal distribution with $\mu = 48.4$ and $\sigma = 7.5$. Approximately what are the probabilities that on any one day, they will receive a) at least 55 complaints; b) anywhere from 40 to 50 complaints? Ans: 0.2090, 0.4933.

C. Normal Approximation to the Binomial Distribution

The normal distribution provides a close approximation to the binomial distribution when *n*, the number of trials, is large, and p, the probability of success on an individual trial is close to ½.

It is considered a sound practice to use the normal approximation to the binomial distribution only when *np* and *nq* are both greater than 5.

From the binomial distribution: $\mu = np$ $\sigma = \sqrt{npq}$

Then the normal distribution has:

$$z = \frac{x - \mu}{\sigma} = \frac{z - (np)}{\sqrt{npq}}$$

Examples:

58. The probability that a patient recovers from a delicate heart operation is 0.90. Of the next 100 patients having this operation, what is the probability that a) between 84 and 95, inclusive, survive? b) fewer than 86 survive? Ans: 0.9514, 0.0668.

59. A pair of dice is rolled 180 times. What is the probability that a total of 7 occurs a) exactly 30 times; b) at least 25 times; c) between 33 and 41 times, inclusive. Ans: 0.0796; 0.8643, 0.2978